A Survey of Constructive Presheaf Models of Univalence

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Introduction

Any formal system for representing mathematics should address the two questions of how to represent *collections* of mathematical objects and how to decide the laws of *identifications* of these objects. These laws of identifications have become quite subtle. While it has been clear for a long time that it is good mathematical practice to identify *isomorphic* algebraic structures [11], or at least to use only notions and facts about algebraic structures that are invariant under isomorphisms, category theory extends this to the notion of *categorical equivalences*[1], which themselves have been generalized to higher forms of equivalences [25]. Voevodsky noticed that, by extending some versions of dependent type theory with one further axiom – the *univalence axiom* – one obtains a formal system in which all notions and operations are automatically invariant under isomorphisms and even under higher notions of equivalence.

At the same time, Voevodsky showed the consistency of this axiom, by giving a very sophisticated model in terms of so-called Kan simplicial sets, which have been used for an abstract presentation of spaces in algebraic topology and homotopy theory. The relevance of homotopy theory was suggested earlier by Grothendieck, who pointed out the analogy between the laws of higher identifications in mathematics and homotopy theory [25]. This model relies heavily on classical logic, while dependent type theory was originally intended as a language for expressing constructive mathematics. It also *a priori* relies on very strong logical principles – ZFC with a countable hierarchy of inaccessible cardinals – which are much stronger than the ones needed for ordinary dependent type theory [1]. Thus, two natural questions were first to describe exactly the proof theoretic strength of the univalence axiom, and second to see if this axiom could be explained in a constructive setting. These two questions have recently been completely elucidated, as part of a general study of constructive presheaf models of dependent type theory (with univalences). The goal of this article is to provide a survey of these results. It is noteworthy that all of these results are developed in a constructive metatheory, and that most of them have been formally checked in systems based on dependent type theory [2; 9; 32; 35].

1. PRESHEAF MODELS OF DEPENDENT TYPE THEORY

1.1. Some notation

Since we want to present the models in a constructive setting, we shall use as meta language a constructive version of set theory, the system $\mathrm{CZFu}_{<\omega}$ [1]. This system has

[1]In this view for instance, the groupoid of all linear orders with a fixed finite number of elements, should be identified with the trivial groupoid with one object and one arrow. Indeed in both groupoids there is exactly one map between any two objects.

a cumulative hierarchy of (a constructive version of) Grothendieck universes $\mathcal{U}_0, \mathcal{U}_1, \dots$ and it is straightforward to represent basic notions of category theory (and in particular presheaves) in this setting. However, it is possible – and this will be important in one application below – to use as a metalanguage an "extensional" version of type theory with a hierachy of cumulative universes, such as the system NuPrl. Actually, presheaf models of type theory recently have been formalised elegantly in this system by M. Bickford [9].

Most definitions can be given for presheaves over an arbitrary base category, which we assume to be in the first universe, $\mathcal{U}_0$. We will denote by $I, J, K, \dots$ the objects of a given base category, and by $f, g, \dots$ its morphisms. We write $1_I : I \to I$ the identity morphism, and $fg : K \to I$ the composition of $f : J \to I$ and $g : K \to J$. A *presheaf* A is then given by a family of sets $A(I)$ and restriction maps $u \mapsto uf,\ A(I) \to A(J)$ for $f : J \to I$, satisfying the laws $u1_I = u$ and $(uf)g = u(fg)$ for $f : J \to I$ and $g : K \to J$.

One will also need to talk about "subpresheaf". This is usually done by introducing $\Omega(I)$, which is the collection of sieves on I. A *sieve* on I is a set S of maps with codomain I such that $fg : K \to I$ is in S whenever $f : J \to I$ is in S and $g : K \to J$. If 1_I is in S, then S is maximal (consists of all maps with codomain I) and will be denoted by 1. In the constructive and predicative setting of $\mathrm{CZFu}_{<\omega}$, we replace Ω by the presheaf Ω_d, where $\Omega_d(I)$ is the set of *decidable* sieves S (i.e., we can decide if a given $f : J \to I$ is a member of S or not). In this way, $\Omega_d(I)$ becomes a set (in the first universe $\mathcal{U}_0$) and not a class. A *subpresheaf* of a presheaf A is given by a map $A \to \Omega_d$, which for each I selects in a coherent way a subset of the set $A(I)$ of shape I defined by the "polyhedron" A.

Any object I of the base category defines a presheaf $\mathsf{Yo}(I)$, *represented by* I. We define $\mathsf{Yo}(I)(J)$ to be the set of maps $J \to I$ and the restriction maps are defined by composition.

1.2. Motivations and Example

Presheaf models can be seen as a generalization of the notion of Kripke model [43]. In [43], some "vivid" terminology from [30] is used: we think of a presheaf A as a "variable domain". For each I we have a set $A(I)$. The maps $f : J \to I$ give us transitions between "stages" I and "later" stages J; and each such transition "restricts" elements in $A(I)$ to elements in $A(J)$ "along" the map f.

The intuition is *temporal*. One early use of presheaves however by Eilenberg and Zilber [20] (indeed this was before the formulation of Kripke models) had some *spatial* intuitions. Thinking of the objects $I, J, \dots$ as basic "shapes", we can think of $A(I)$ as a given set of objects of shape I. Eilenberg and Zilber suggested then an elegant abstract and combinatorial representation of spaces as "complexes". The idea is to define the notion of "polyhedron" as a presheaf on a base category of given "shapes".[2] The notion of "subpolyhedra" is then represented by subpresheaves,

A particular case is given by the presheaf of *simplicial sets*. The base category is the category finite linear posets $[n] = \{0, \dots, n\}$ and monotone maps. We write Δ^n for the presheaf represented by $[n]$. In particular Δ^1 can be seen as an abstract representation of the unit interval with two distinct global elements 0 and 1. We can think of Δ^n as an abstract representation of an n dimensional tetrahedron. The basic shapes are then points, lines, triangles, tetrahedra, and so on.

[2]For Eilenberg and Zilber, the shapes were tetrahedra. This representation of complexes by presheaves was used by D. Kan with two different notions of shapes: in [28] they are cubes, and in [29] they are tetrahedra.

1.3. Language of dependent type theory for presheaf models

Dependent type theory has a natural semantics in presheaf models [35; 32; 26]. We will only use this language in an informal way, and recall briefly the main operations. This section can be seen as a generalization to the language of dependent types of the semantics of simply typed lambda-calculus presented in the seminal paper of D. Scott [43].

We work with presheaves on a given category. A *type* is interpreted as a presheaf A: a family of sets $A(I)$ with restriction maps $u \mapsto uf,\ A(I) \to A(J)$ for $f : J \to I$. A dependent type B on A is interpreted by a presheaf on the category of *elements* of A: the objects are pairs I, u with u in $A(I)$ and morphisms $f : (J, v) \to (I, u)$ are maps $f : J \to I$ such that $v = uf$. A *dependent type* B is thus given by a family of sets $B(I, u)$ and restriction maps $B(I, u) \to B(J, uf)$.

We think of A as a type and a dependent type B over A as a family of presheaves $B(x)$ varying with x in A. An element u in $A(I)$ is thought of as an element of A defined at "stage" I, and we can write $B(x)(x = u)$ instead of $B(I, u)$.

It is straightforward to define the operation $\Sigma(x : A)B(x)$. The set $(\Sigma(x : A)B)(I)$ is the set of pairs u, v with u in $A(I)$ and v in $B(I, u)$ and the restriction map is $(u, v)f = (uf, vf)$.

The operation $\Pi(x : A)B(x)$ generalizes the semantics of implication in a Kripke model. An element w of $(\Pi(x : A)B(x))(I)$ is a family of functions w_f in $\Pi(u \in A(J))B(J, u)$ for $f : J \to I$ such that $(w_f\, u)g = w_{fg}\,(ug)$ if u in $A(J)$ and $g : K \to J$. A particular case is the exponential D^A of two presheaves, since any presheaf D can be considered as a constant family of types $\overline{D}$ over A defined by $\overline{D}(I, u) = D(I)$.

An example of presheaf is given by Ω_d where $\Omega_d(I)$ is the set of decidable sieves on I. We have a dependent type $[\psi]$ for $\psi : \Omega_d$ where $[\psi](\psi = S)$ for S in $\Omega_d(I)$ is $\{\mathsf{tt} \mid 1_I \in S\}$, where tt is a fixed object. The type $[\psi]$ is a subsingleton: if we have a and b of type $[\psi]$ then $a = b$. The unfolding of this statement is the following: if S is a (decidable) sieve on I and a, b are elements in $\{\mathsf{tt} \mid 1_I \in S\}$ then $a = b$. This is indeed the case (even if the sieve S is not decidable) since then both a and b are equal to tt.

If A is a presheaf and $\psi : A \to \Omega_d$ we can form the subpresheaf $A|\psi$ of elements a in A such that ψa holds. The unfolding of this operation is that $(A|\psi)(I)$ is the set of elements u in $A(I)$ such that 1_I is in the sieve $\psi\, u$.

If A is a presheaf, we can form $A^{[\psi]}$ for $\psi : \Omega_d$. For S a (decidable) sieve on I it can be checked that $(A^{[\psi]})(\psi = S)$ is canonically isomorphic to the set of partial elements of A defined on S: the set of families u_f in $A(J)$ for $f : J \to I$ such that $u_f g = u_{fg}$ if $g : K \to J$. It is quite suggestive to see such a partial element u as a map from the subsingleton $[\psi]$ into A. For instance, any a in A defines the element $u = \lambda(x : [\psi])a$. This corresponds to the fact that any element a in $A(I)$ defines a family $u_f = af$ in $A(J)$ for $f : J \to I$ in S, which satisfies $u_f g = u_{fg} = afg$ for $g : K \to J$.

We can use this to associate to any presheaf A the presheaf $|A|$ of "partial elements" of A, by defining $|A| = \Sigma(\psi : \Omega_d)A^{[\psi]}$. An element of $|A|(I)$ is given by a sieve S on I and a family of elements u_f in $A(J)$ for $f : J \to I$ in S such that $u_f g = u_{fg}$ if $g : K \to J$. If ψ, u is a partial element of A, we call ψ the *extent* of this partial element. Any element a in A defines a "total" element $(1, \lambda(x : [1])a)$ of extent 1.

Working in the system $\mathrm{CZFu}_{<\omega}$, we have a hierarchy of universes $\mathcal{U}_n$ in the underlying set theory. We can then define $U_n(I)$ to be the set of $\mathcal{U}_n$ presheaves over $\mathsf{Yo}(I)$. An element of $U_n(I)$ is given by a collections of sets $A(J, f)$ in $\mathcal{U}_n$ for $f : J \to I$ with restriction maps $A(J, f) \to A(K, fg)$ for $g : K \to J$. In general, a type at "stage" I is given by a family of sets $A(J, f)$, without restriction of sizes, for $f : J \to I$ with restriction maps $A(J, f) \to A(K, fg)$. We can then define a partial type of size n to be an element in $|U_n|$.

Concretely, such an partial type at stage I is given by a sieve S on I and a family of sets $A(J,f)$ in $\mathcal{U}_n$ for $f : J \to I$ in S with restriction maps $A(J,f) \to A(K,fg)$ for $g : K \to J$.

2. MODELS OF UNIVALENCE

We now explain how to build a model of type theory with the univalence axiom inside a presheaf model given two presheaves satisfying some special conditions. This step is similar to an inner model construction in set theory. This is inspired from Voevodsky's simplicial set model, but carried out in a a constructive metatheory. As we shall see, this approach requires a special property of the interval (the interval should be "tiny"), which does not hold for simplicial sets. (The usual description of the simplicial model cannot be carried out in a constructive framework, as shown [7; 36], and it is not known at this point if there is another constructive description of this model.)

The model is parametrised by two special presheaves: a subpresheaf $\mathbb{F}$ of Ω_d (which will classify *cofibrant* maps) and a presheaf $\mathbb{I}$ which is a formal representation of the interval. The subpresheaf $\mathbb{F}$ corresponds to a map $\mathsf{Cofib} : \Omega \to \Omega$, so that $\mathbb{F}$ can be seen internally as the subpresheaf of propositions ψ in $\mathbb{F}$ satisfying the property $\mathsf{Cofib}(\psi)$ (which reads "ψ is cofibrant").

2.1. Properties of Cofib

The axioms for Cofib, as isolated in [35], are reminiscent of the ones used for synthetic topology [21] and synthetic domain theory [40] (with only formal connections at this point however). These are

— (A_1) The predicate Cofib should define a *dominance*, i.e. cofibrant maps should contain isomorphisms and be closed under composition, which can be expressed by $\mathsf{Cofib}(1)$ and $\mathsf{Cofib}(\exists(u : [\varphi])\psi u)$ whenever $\mathsf{Cofib}(\varphi)$ and $\mathsf{Cofib}(\psi u)$ for $u : [\varphi]$.
— (A_2) $\mathbb{F}$ is closed under disjunction: $\mathsf{Cofib}(\psi_1 \vee \psi_2)$ if $\mathsf{Cofib}(\psi_1)$ and $\mathsf{Cofib}(\psi_2)$.

2.2. Properties of $\mathbb{I}$

— (B_1) The presheaf $\mathbb{I}$ has two global elements 0 and 1 that are distinct (i.e. we have $\neg(0 = 1)$ in the internal logic [43] of the presheaf model)
— (B_2) $\mathbb{I}(J)$ has a decidable equality for each object J
— (B_3) $\mathbb{I}$ is *tiny*, i.e. the path functor $X \mapsto X^{\mathbb{I}}$ has a right adjoint
— (B_4) $\mathbb{I}$ has connections

The last condition means that we have two binary operations $\wedge$ and $\vee$ on $\mathbb{I}$ satisfying $x \wedge 0 = 0 \wedge x = 0$ and $x \wedge 1 = 1 \wedge x = x$ and $x \vee 0 = 0 \vee x = x$ and $x \vee 1 = 1 \vee x = 1$. These operations can be seen as formal minimum and maximum operations.

This presheaf $\mathbb{I}$ is used to represent the type of paths, so that if a_0 and a_1 are in A, then $\mathsf{Path}\ A\ a_0\ a_1$ is the type of elements $\omega : A^{\mathbb{I}}$ such that $\omega(0) = a_0$ and $\omega(1) = a_1$.

Note that the property (B_2) implies that the equality $i = j$ is in Ω_d for all i and j in $\mathbb{I}$.

The condition (B_3) appears in the setting of synthetic differential geometry [30] (but there also, the connections may be only formal), and plays a crucial rôle in defining the universe of "fibrant" types.

2.3. Conditions mixing $\mathbb{I}$ and $\mathbb{F}$

— (C_1) For all i in $\mathbb{I}$ we have $\mathsf{Cofib}(i = 0)$ and $\mathsf{Cofib}(i = 1)$.
— (C_2) We have $\mathsf{Cofib}((\forall i : \mathbb{I})\psi)$ if $\forall(i : \mathbb{I})\mathsf{Cofib}(\psi)$.

The second condition appears in the setting of synthetic topology and expresses that $\mathbb{I}$ is *compact* [21] (which is appropriate since $\mathbb{I}$ is supposed to represent the unit in-

(A$_1$) Cofib should define a *dominance*
(A$_2$) Cofib is closed by disjunction
(B$_1$) The presheaf $\mathbb{I}$ has two global elements 0 and 1 that are distinct
(B$_2$) $\mathbb{I}(J)$ has a decidable equality for each object J
(B$_3$) $\mathbb{I}$ is *tiny*
(B$_4$) $\mathbb{I}$ has connections
(C$_1$) $\forall(i : \mathbb{I})\ \mathsf{Cofib}(i = 0) \wedge \mathsf{Cofib}(i = 1)$
(C_2) $\mathsf{Cofib}((\forall i : \mathbb{I})\psi)$ if $\forall(i : \mathbb{I})\mathsf{Cofib}(\psi)$

Fig. 1: Conditions on $\mathbb{I}$ and Cofib

terval). Another formulation of this condition (noticed by Ch. Sattler) is that the path functor $X \mapsto X^{\mathbb{I}}$ preserves cofibrations.

These conditions are collected in figure 2.3.

2.4. Some laws of identification and fibrations

Our method to build this model is to interpret the equality type as the path type, defined above using the interval $\mathbb{I}$. For instance, the reflexivity of equality holds, since we have the constant path $1_a = \lambda(i : \mathbb{I})a$ which is element of type $\mathsf{Path}\ A\ a\ a$ for any a in A.

Another law of identification that is valid at the abstract level and that uses the connections on the interval (B$_4$) is that for any type A and any element a in A the type

$$T = \Sigma(x : A)\mathsf{Path}\ A\ a\ x$$

satisfies the following property: we have an element $(a, 1_a)$ in T and there is a path from this element to any other element (x, ω) in T. Indeed the path $\theta = \lambda(i : \mathbb{I})(\omega(i), \lambda(j : \mathbb{I})\omega(i \wedge j))$ satisfies $\theta(i) : T$ for all i and $\theta(0) = (a, 1_a)$ and $\theta(1) = (x, \omega)$.

To have such a path is fundamental both in type theory [34], and in homotopy theory, since it was exactly this basic fact that, according to J.-P. Serre himself [45], was at the origin of the loop space method in homotopy theory [44].

The problem comes with the principle of "substituting equal for equal", expressed in type theory. If A is a dependent type over Γ, it should be the case that given a path γ in $\Gamma^{\mathbb{I}}$ we should at least have a transport function $A\gamma(0) \to A\gamma(1)$. Such a function does not need to exist however in general[3].

We then try to find an operation at least as strong as having a transport, and which is furthermore closed under all operations of our type theory: dependent products and sums, path types, and the universes.

Such an operation is the following. It can be seen as a subtle refinement of the path lifting property (any path in Γ can be lifted to a path in A), and has been isolated in homotopy theory [18].

Definition 2.1. A *filling operation* for a dependent type A over a context Γ is an operation which, given a path γ in $\Gamma^{\mathbb{I}}$ and a cofibrant proposition ψ in $\mathbb{F}$, extends any partial section in $\Pi(i : \mathbb{I})[\psi \vee i = 0] \to A\gamma(i)$ or in $\Pi(i : \mathbb{I})[\psi \vee i = 1] \to A\gamma(i)$ to a total section in $\Pi(i : \mathbb{I})A\gamma(i)$.

Note that we can define internally a type $\mathsf{Fill}(\Gamma, A)$ of all filling operations for A.

If $\psi = 0$ then we get exactly the two path lifting operations, which given a path γ in the base Γ and a starting (resp. ending) point in $A\gamma(0)$ (resp. in $A\gamma(1)$) lifts this path to a path in $\Pi(i : \mathbb{I})A\gamma(i)$.

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[3]For instance, if we take $\Gamma = \mathbb{I}$ and $\gamma = 1_{\mathbb{I}}$, and we define $A(J, r) = 1$ for $r = 0$ and $A(J, r) = \emptyset$ if $r \neq 0$ then there is no function of type $A\gamma(0) \to A\gamma(1)$.

This refinement can be seen as a formulation of the following principle of *homotopy extension property* emphasized by Eilenberg [18; 19][4].

PROPOSITION 2.2. *If A is a subpolyhedron of B, given two homotopic functions $f_0\ f_1 : A \to X$ and an extension $f_0' : B \to X$ of f_0, there is an extension f_1' of f_1 homotopic to f_0.*

In the case Γ is the unit context 1, the type A is a global type and an element of $\mathsf{Fill}(1, A)$ extends a partial section in $\Pi(i : \mathbb{I})A^{[\psi \vee i=0]}$ (resp. $\Pi(i : \mathbb{I})A^{[\psi \vee i=1]}$) to a path in $A^{\mathbb{I}}$. We let $\mathsf{Fib}(A)$ be the type of such extension operations. Note that we cannot identify $\mathsf{Fill}(\Gamma, A)$ with $\Pi(\rho : \Gamma)\mathsf{Fib}(A\rho)$: to be pointwise fibrant is in general weaker than to have a filling operation[5]. The filling operation is expressed as a type over the path context $\Gamma^{\mathbb{I}}$ and not over the context Γ.

We define a *composition operation* for a dependent type A over a context Γ as an operation c_A which, given a path γ in $\Gamma^{\mathbb{I}}$, a cofibrant proposition ψ in $\mathbb{F}$ and a partial section u in $\Pi(i : \mathbb{I})[\psi \vee i = 0] \to A\gamma(i)$ (resp. $\Pi(i : \mathbb{I})[\psi \vee i = 1] \to A\gamma(i)$), produces an element $c_A\ \gamma\ \psi\ u$ in $A\gamma(1)$ (resp. $A\gamma(0)$) such that $\psi \Rightarrow u\ 1\ \mathsf{tt} = c_A\ \gamma\ \psi\ u$ (resp. $\psi \Rightarrow u\ 0\ \mathsf{tt} = c_A\ \gamma\ \psi\ u$). Let us write $\mathsf{Comp}(\Gamma, A)$ the type of all composition operations on A. Any filling operation f_A defines a composition operation by taking $c_A\ \gamma\ \psi\ u = f_A\ \gamma\ \psi\ u\ 1$ (resp. $c_A\ \gamma\ \psi\ u = f_A\ \gamma\ \psi\ u\ 0$) and this provides a map $\mathsf{Fill}(\Gamma, A) \to \mathsf{Comp}(\Gamma, A)$. Using connections (property B_4 of the interval) we can show the following.

PROPOSITION 2.3. *The map $\mathsf{Fill}(\Gamma, A) \to \mathsf{Comp}(\Gamma, A)$ has a section.*

This is convenient since it is sometimes easier to define a composition operation than a filling operation (this will be the case for dependent products and for universes).

We can define a *contractibility* structure $\mathsf{Contr}(T)$ of a type T. An element of $\mathsf{Contr}(A)$ is an extension operation that extends any partial element in $\Sigma(\psi : \mathbb{F})T^{[\psi]}$ to a total element. A *contractibility structure* for a dependent type A over Γ is then an element of $\Pi(\rho : \Gamma)\mathsf{Contr}(A\rho)$. (Thus, in contrast, with the filling operation, a contractibility operation for a family over Γ is expressed by a type over Γ itself.)

Before dealing with the representation of universes, there is a further operation that can be expressed internally, and that plays a crucial rôle in showing that the universes themselves are fibrant and univalent.

PROPOSITION 2.4. *We can build an operation of type $\Pi(A : U_n)\mathsf{Contr}(\Sigma(T : U_n)(T \to A))$ which takes a partially defined function $T : U_n,\ u : T \to A$ with codomain A and extends it to a totally defined function.*

2.5. Universes of fibrant types

We now define a universe of fibrant types. It should be a global object U_n such that there is a natural isomorphism between the set of maps $\Gamma \to \mathsf{U}_n$ and the set of pairs A, f_A where $A : \Gamma \to U_n$ and f_A is an element of $\mathsf{Fill}(\Gamma, A)$. (This is expressed by the fact that U_n should *classify* the family of types over Γ with a given filling structure.) It would be wrong to define it as $\Sigma(X : U_n)\mathsf{Fib}(X)$ since this would only classify families of types that are pointwise fibrant. The problem comes essentially from the fact that the filling operation for a family A over Γ is expressed by a type $T(A)$ over $\Gamma^{\mathbb{I}}$ and not over a type over Γ. While analysing how this definition works in some special cubical model we found with Ch. Sattler [17] that a sufficient condition for this is the condition

[4]In the Bourbaki's notes on homotopy by Eilenberg, 1951, it is written that proofs of basic results about homotopy "can be obtained quite neatly by repeated, and sometimes tricky, use of this general lemma".

[5]For instance if we define a dependent type A over $\mathbb{I}$ by taking $A(J, r) = \emptyset$ if $r \neq 0$ and $A(J, r) = 1$ if $r = 0$ then $\mathsf{Fill}(\mathbb{I}, A)$ is empty, since we don't have a transport function $A0 \to A1$, but we do have $\Pi(\rho : \mathbb{I})\mathsf{Fib}(A\rho)$.

(B_3) on the interval: the path functor $X \mapsto X^{\mathbb{I}}$ has a right adjoint. This is equivalent to the fact that this functor preserves small colimits and it implies a dependent right adjoint operation: given T over $\Gamma^{\mathbb{I}}$ we can find $R_d(T)$ over Γ with a natural isomorphism between $\mathsf{Elem}(\Gamma, R_d(T))$ and $\mathsf{Elem}(\Gamma^{\mathbb{I}}, T)$. It is then possible to express the filling operation as the type $C(A) = R_d(T(A))$ over Γ, and this filling operation then can be seen as a map $C : U_n \to U_n$. The universe of fibrant types (of size n) can be defined simply as $\Sigma(X : U_n)C(X)$, and we can re-express the closure operations as operations of types

$$\Pi(A : U_n)\Pi(B : A \to U_n)C(A) \to (\Pi(x : A)C(B\ x)) \to C(\Pi\ A\ B)$$
$$\Pi(A : U_n)\Pi(B : A \to U_n)C(A) \to (\Pi(x : A)C(B\ x)) \to C(\Sigma\ A\ B)$$
$$\Pi(A : U_n)C(A) \to \Pi(a_0\ a_1 : A)C(\mathsf{Path}\ A\ a_0\ a_1)$$
$$C(\Sigma(x : U_n)C(X))$$

To build these operations is actually straightforward, except for the universe; the only subtle point is closure under dependent products, which relies on the connection and Proposition 2.3. For the universe, this also relies on Proposition 2.3 and on a refinement of Proposition 2.4 ("equivalence extension operation") which expresses that the extension provided by Proposition 2.4 of an equivalence is an equivalence. It is exactly (and only) at this point that we need to use the property (C_2) of the interval (the fact that the interval is "compact").

Using these operations, we can build a new model of type theory where a type is given by a pair A, c_A of a presheaf A and an element of type $C(A)$. An element of this type is simply an element of A. This model satisfies equivalence.

The definition of universes we have presented cannot be done internally, in contrast with the description of all other operations of type theory (dependent products and sums, and path). Indeed, it is crucial to express the existence of the right adjoint of the path functors externally, since to express it internally[6] leads to a contradiction [32].

If we extend type theory with suitable *modalities*, it is however possible to express in this extension that some facts hold only at a *global* level. This is what has been achieved in the work [32], which could in this way checks formally the correctness of the definition of universes. Together with the previous work [35], this provides a formal check of the correctness of presheaf models of type theory with the univalence axiom from the conditions listed in figure 2.3.

2.6. Identity types

The models we have presented so far model exactly the rules of identity elimination as presented by Martin-Löf [34]. The problem essentially is that these rules express that the path lifting of a constant path has to be constant, but there is nothing that enforces this in our presentation.

This problem was solved by A. Swan [46] using ideas from homotopy theory. We present here an adaptation of this idea, which becomes especially simple in the kind of models we consider. We define $\mathsf{Id}\ A\ a\ b$ to be the type of pairs ψ, ω with ψ in $\mathbb{F}$ and ω in $\mathsf{Path}\ A\ a\ b$ such that $\psi \Rightarrow \mathsf{const}(\omega)$ where $\mathsf{const}(\omega)$ is defined to be $\forall(i : \mathbb{I})\omega(i) = \omega(0)$ and expresses that ω is constant. The reflexivity proof $1, \lambda(i : \mathbb{I})a$ is then an element in $\mathsf{Id}\ A\ a\ a$. If ψ, ω is in : $\mathsf{Id}\ A\ a\ b$ and we have a family of type P over A with an element in $\mathsf{Fill}(A, P)$ and $u : P\ a$ then we can use the element in $\mathsf{Fill}(A, P)$ to extend the map

$$w : \Pi(i : \mathbb{I})[\psi \vee i = 0] \to P(\omega\ i) \qquad w\ i\ x = u$$

to a section s in $\Pi(i : \mathbb{I})P(\omega\ i)$ that satisfies $\psi \Rightarrow s\ 1 = u$. With this simple modification, we get an interpretation of all rules of identity types as introduced by Martin-Löf [34].

[6]In the form: we have $R : U_n \to U_n$ with a canonical isomorhism between $A^{\mathbb{I}} \to B$ and $A \to R(B)$.

2.7. Examples

The right adjoint condition (B_3) is satisfied as soon as $\mathbb{I}$ is representable and the base category has binary products. This includes the Lawvere category of distributive lattices, or de Morgan algebras [14], or even of Boolean algebras. We get in this way a large class of models of type theory with univalence. Such Lawvere categories have a syntactical presentation, and it is then possible to give a syntactic presentation of the corresponding models of type theory, as done e.g. in [14]. (This point was emphasized in Voevodsky's talk [52].) An important result relating these models is the canonicity result [27]: any closed term in Nat (maybe using univalence) is convertible to a numeral.

An example where all conditions hold *except for* the condition (B_3) is provided by simplicial sets with the canonical choice $\mathbb{I} = \Delta^1$. If the path functor $X \mapsto X^{\mathbb{I}}$ had a right adjoint, then it would preserve colimits, but it can be checked that this is not the case for $\mathbb{I} = \Delta^1$.

2.8. Variations

A possible variation is to take away the condition B_4 on the interval (connections) and to strengthen the condition C_1 as $\forall(i\ j : \mathbb{I})\mathsf{Cofib}(i = j)$. The notion of filling structures for a type A over Γ is then changed to an operation that, given γ in $\Gamma^{\mathbb{I}}$ and k in $\mathbb{I}$ and ψ in $\mathbb{F}$, extends any partial section in $\Pi(i : \mathbb{I})[\psi \vee i = k] \to A\gamma(i)$ to a total section in $\Pi(i : \mathbb{I})A\gamma(i)$.

Then it is possible to show that with these changes, we still get a model of type theory with a cumulative hierarchy of univalent universes. The fact that we get univalent universes in this way was discovere in [5]. This has even been formally checked in [2]. A type system in the style of NuPrl corresponding to this model is developed in [3; 4; 5].

2.9. Refinement

The model extends directly to inductive data types. For instance the type $\mathsf{W}(x : A)B$ which has for constructors $\mathsf{sup}\ a\ u$ with a in A and u in $B(a) \to \mathsf{W}(x : A)B$ has a direct interpretation in presheaf models, and it can be checked that we have

$$C(A) \to (\Pi(x : A)C(B)) \to C(\mathsf{W}(x : A)B)$$

The paper [15] explains how to extend the semantics to higher inductive types [48]. (Note that it is not clear at this point if a semantics of higher inductive types can be given in the simplicial set model. Maybe this is possible using the techniques presented in [15].)

3. APPLICATIONS

3.1. Application 1: proof theoretic strength

As explained above, we can take as meta language the system CZF extended with a hierarchy of universes [1], or the system used for the proof system NuPrl. (For the latter choice, it should be mentioned that M. Bickford has actually checked the correctness of the cubical type theory presented in [14] in the system NuPrl.) These systems are known to be of the same proof theoretic strength as pure dependent type theory with dependent products, sums, universes and W-types (and *no* identity types) [1]. Hence, we can state:

THEOREM 3.1. *The addition of axiom of univalence and higher inductive types does not increase the proof theoretic strength of dependent type theory.*

It follows e.g. that the provably total functions $\mathbb{N} \to \mathbb{N}$ are the same in all these systems.

3.2. Application 2: impredicative universes

Instead of NuPrl, we could work in an *extensional* type theory (with the so-called equality reflection rule) and with an *impredicative* universe, i.e. a universe U that satisfies $\Pi(x : A)B$ is of type U if B is of type U for x in A, for *any* type A. It is known how to build models of such a system [33]. Starting from such a type theory, we obtain a model of a type theory with an impredicative *univalent* universe. This was achieved recently in the work [49].

It then is possible to give a semantics model for the various direct elegant impredicative definitions of higher inductive types described in [6]. For instance, while it is well-known since the work of Reynolds [37] that we can represent the type of natural numbers as a subtype of $\Pi(X : U)(X \to X) \to (X \to X)$, one can represent the *circle* in a similar way as a subtype of $\Pi(X : U)\Pi(b : X)(\mathsf{Id}\ X\ b\ b) \to X$. One also can represent the type of *integers* as a subtype of $\Pi(X : U)(\mathsf{Equiv}\ X\ X) \to (X \to X)$.

3.3. Application 3: independence and consistency results

We have explained how to build presheaf models of type theory, starting from a base category $\mathcal{C}$ and presheaves $\mathbb{I}$ and $\mathbb{F}$ satisfying the conditions of figure 2.3. As noticed in [35], Corollary 8.5, these conditions are always satisfied if the base category has finite products, $\mathbb{I}$ is representable and we take $\mathbb{F} = \Omega_d$. This is the case if the base category $\mathcal{C}$ is the Lawvere theory of distributive lattice for instance.

In this way, we obtain a model of type theory with a cumulative hierarchy of univalent universes (and higher inductive types [15]).

Given another category with finite product $\mathcal{D}$, if we change the base category with $\mathcal{C} \times \mathcal{D}$ and redefine the interval as $\tilde{\mathbb{I}}(J, X) = \mathbb{I}(J)$, still taking cofibrant maps classified Ω_d, we get a new model of type theory which can be thought of as a Kripke model, with worlds as objects of $\mathcal{D}$, over the previous model.

The interest of this operation is that we can have a site structure on the category $\mathcal{D}$ and this site structure can give rise to a non trivial (internal) family of *left exact modalities* as considered in the paper [38]. The objects modal for all these modalities [38] form then a new model of type theory with univalences. This can be seen as generalizing the internal description of *sheaves* in a presheaf model.

For instance, if we take for the base category the poset of clopen (simultaneously closed and open) subsets of Cantor space, with its canonical notion of covering, we obtain a generalization of the model presented in [16], and we obtain the following result.

THEOREM 3.2. *There is a model of type theory with a hierarchy of univalent universes which validates the* negation *of Markov's Principle.*

Using the space $[0, 1]$ as in [16], we get similarly.

THEOREM 3.3. *There is a model of type theory with a hierarchy of univalent universes and higher inductive types which validates the* negation *of countable choice.*

Finally, using the opposite of the category of Boolean algebras with decidable equality we obtain:

THEOREM 3.4. *There is a model of type theory with a hierarchy of univalent universes and higher inductive types which validates Brouwer's principle of uniform continuity for functions* $2^{\mathbb{N}} \to \mathbb{N}$.

3.4. Application 4: Quillen model structures

Given a presheaf model with presheaves $\mathbb{I}$ and $\mathbb{F}$ satisfying the conditions of figure 2.3, we have a class of *cofibrations*, that are the maps classified by $\mathbb{F}$. Any map $u : A \to B$ can be seen as defining a dependent type F over B, by taking $F(I, b)$ for b in $B(I)$ to be the set of elements a in $A(I)$ such that $u\ a = b$. We define this map to be a *fibration* if we have a global element in $\mathsf{Fill}(B, F)$. This can be expressed as a right lifting property, in the style of abstract homotopy theory as presented in [13; 39; 24; 22]. Similarly such a map u is defined to be a *trivial fibration* if we have a global element in $\mathsf{Contr}(B, F)$ or, equivalently, the map u has the right lifting property w.r.t. any cofibration.

Using the property A_1 of Cofib (dominance) one can for instance show the following:

PROPOSITION 3.5. *A map is a cofibration if, and only if, it has the left lifting property w.r.t. any fibration.*

Using the techniques explained in [47; 15], it is possible to factor any map u in the form $u = pi$ where i is a map having the left lifting property w.r.t. any fibration and p is a fibration. Following [41], we can define the map u to be a *weak equivalence* if the map p is a *trivial* fibration. Using in a crucial way the fact that we have fibrant universes of fibrant types, we can then, following Section 2 of [41], show that the three class of maps *fibrations, cofibrations, and weak equivalences* that we have defined form a *Quillen model structure* [39] on the presheaf category. Note that these arguments can all be carried out in a constructive meta theory.

In a classical framework, we can take $\mathbb{F} = \Omega$ and it is interesting to compare this model structure with the so-called "Cisinski" model structure [13] that we obtain from a presheaf $\mathbb{I}$ having two distinct global objects. These two model structures have the same fibrant objects and same cofibrant maps. It follows then from a result of A. Joyal (presented e.g. in [39], Theorem 15.3) that these two model structures coincide. As an application, we can state (which can be seen as an application of dependent type theory to abstract homotopy theory):

THEOREM 3.6. *Given any presheaf model with an interval $\mathbb{I}$ satisfying the conditions of figure 2.3 (with $\mathbb{F} = \Omega$) the corresponding Cisinksi model structure is* complete *in the sense of [13].*

In the case where the base category is the Lawvere theory of distributive lattices or de Morgan algebras, there is a canonical geometric realization map from presheaves to topological spaces, which associates to $\mathbb{I}$ the real unit interval $[0, 1]$. A natural question is whether this functor sends a weak equivalence to a weak homotopy equivalence [24; 39]. Ch. Sattler has shown that this is *not* the case for de Morgan algebras: if we take the quotient L of $\mathbb{I}$ by the involution of de Morgan algebra, the geometric realization of L is a contractible space, but the map $L \to 1$ is not a weak equivalence. An important open problem is whether this holds in the case of distributive lattices. Maybe it is not possible to capture in a constructive framework the notion of equivalence of the classical model structure on topological spaces.

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